TOPOLOGY AND QUANTIZATION [*]

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Abstract

A simple algebraic model of charged particle coupled to singular magnetic field is given. Quantization is described as gradation by certain abelian group G. Statistics is determined by a commutation factor λ on the grading group G. Composite fermions and composite bosons are described in an unified way as $Z_2 \oplus ... \oplus Z_2$-graded λ-commutative algebras.

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1 Introduction

It is well-known that there is a mathematical formalism for the description of particle systems in a low dimensional space $\mathcal{M}$ based on the notion of the braid group $B_n(\mathcal{M})$, see Ref.[1, 2] for example. In this this approach the configuration space for the system of n–identical particles moving on a manifold $\mathcal{M}$ is $Q_n = (\mathcal{M}^{\times n} - D)/S_n$, where D is the subcomplex of the Cartesian product $\mathcal{M}^{\times n}$ on which two or more particles occupy the same position and S_n is the symmetric group. The group $\pi_1(Q_n(M)) \equiv B_n(M)$ is known as the n–string braid group on $\mathcal{M}$. The statistics of the given system of particles is described by the group $\Sigma_n(\mathcal{M})$ which is a subgroup of the braid group B_n corresponding to interchanges of two particles, one by another. Quantizations are described by unitary representations of the braid group. This picture breaks upin the one-dimensional case. The difficulty arises with the proper definition of the group $\Sigma_n(\mathcal{M})$, see [3]. In this paper we going to present a proposal for quantum model of charged particle in low-dimensional space $\mathcal{M}$ with perpendicular singular magnetic field. Our model is pure algebraic. It is based on the notion of the homotopy theory and G-graded structures. Our fundamental assumption is that every charged particle such as electron coupled to the singular magnetic field is transformed into a composite system which consists a charge and certain number of magnetic fluxes. Every magnetic flux can be compensated by the particle such that the effective magnetic field is zero or can not be compensated. In the first case we say that we have a quasiparticle state, in the second - a quasihole one. We also assume that there is in average N fluxes per particle. This means that the filling factor is $v = \frac{1}{N}$. If the number N of fluxes is even, then the system can be identified as composite fermions. If N is odd, then we obtain composite bosons. The paper is organized as follows. In Sect. 2 we study the quantizations of our particle moving in the singular magnetic field. Note that the "effective" configuration space for our particle is $\mathcal{M} \equiv S^1 \times \ldots \times S^1$. The fundamental group of $\mathcal{M}$ is denoted by $\pi_1(\mathcal{M}, m_0)$, the base point m_0 is the initial point of our particle in the moment $t = 0$. Let G be certain subgroup of the fundamental group, then there is a group algebra $H := \mathbb{C}G$ which has a Hopf algebra structure. In our algebraic approach all possible quantizations are described by H-comodule coaction on a given Hilbert space E. It is known that H-coaction on E is equivalent to the G-gradation of E, see [4]. Hence quantization can be also described as certain gradation of E.

The statistics is determined by the normalized bicharacter (a commutation factor) λ on the grading group G, i.e. a mapping $\lambda : G \times G \longrightarrow \mathbb{C} \setminus \{0\}$ satisfing some condition given in Sec. 2. The algebra of quantum states for our composite system of particle and magnetic fluxes is described in general case as G-graded λ-commutative algebra in Sec. 3, where G is an arbitrary abelian group and λ is a commutation factor on it. In Sec. 4 the algebra of states for composite fermions or composite bosons is described in more details as G-graded λ-commutative algebra $\mathcal{A}_\lambda$ which is generated by x_i, $(i = 1, ..., N)$ such that we have the following commutation relations

$$x_i \; x_j = \lambda_{ij} \; x_j \; x_i, \tag{1}$$

where the factor $\lambda = \lambda_{ij}$ is defined by the following formula

$$\lambda_{ij} = -(-1)^{\delta_{ij}} \; q^{\Omega_{ij}} \tag{2}$$

for $i, j = 1, 2, ..., N$, $\Omega_{ij} = 1$ for $i < j$, $\Omega_{ij} = -\Omega_{ji}$, $q = exp(\pi i N)$ and the grading group is $G \equiv Z_2 \oplus ... \oplus Z_2$ (N-sumands). Such algebra is realized as certain subalgebra in the tensor product of paragrassmann (or parabose) algebra by Clifford algebra. Some physical consequences of our model are given. Note that similar graded structures has been studied previously by several authors but in different context, see Ref. [6, 7, 8] for example. The paper is continuation of the author's previous ones [12, 13, 14, 15].

2   Quantization as gradation

Let us consider an arbitrary charged particle moving on the space $\mathcal{M}$. We assume that $\mathcal{M}$ is path-connected topological with base point m_0. All possible classical trajectories of the particle are path in $\mathcal{M}$. We denote by $P_m\mathcal{M}$ the space of all homotopy classes of paths which starts at m_0 and and end at arbitrary point $m \in \mathcal{M}$. It is known that the union $\cup_m P_m\mathcal{M} = P\mathcal{M}$ is a covering space $P = (P\mathcal{M}, \pi, \mathcal{M})$. The projection $\pi : P\mathcal{M} \longrightarrow \mathcal{M}$ is given by

$$\pi(\xi) = m \quad \text{iff} \quad \xi \in P_m\mathcal{M}. \tag{3}$$

The homotopy class $P_{m_0}\mathcal{M}$ of all paths which start at m_0 and end at the same point m_0, (i.e. a loop space) can be naturally endowed with a group structure. This group is known as the fundamental group of $\mathcal{M}$ at m_0 and

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is denoted by $\pi_1(\mathcal{M}, m_0)$. Generators of the fundamental group are denoted by σ_i. In our case we have $\mathcal{M} = S^1 \times \ldots \times S^1$ and the fundamental group is

$$\pi_1(\mathcal{M}, m_0) = Z \oplus \ldots \oplus Z \quad \text{(N-sumands)}. \tag{4}$$

Let G be a subgroup of the fundamental group $\pi_1(\mathcal{M}, m_0)$. We assume that quantum states of magnetic field can be represented as linear combinations of elements (over a field $\mathbb{C}$ of complex numbers) of the group G. It is known such that linear combinations of elements of certain group G form a group algebra $H := \mathbb{C}G$. It is also known that there is a Hopf algebra structure defined on the group algebra H. Let us denote by E a Hilbert space of quantum states of particle which is not coupled to magnetic field. Quantum states of particle coupled to our singular magnetic field is described by the tensor product $E \otimes H$. Every attaching of magnetic fluxes to the particle moving in singular magnetic field can be represented by certain coaction ρ_E of the Hopf algebra H on the space E, i.e. by a linear mapping

$$\rho_E : E \longrightarrow E \otimes H, \tag{5}$$

which define a (right-) H-comodule structure on E. Note that if E is a H-comodule, where $H = \mathbb{C}G$, then E is also a G-graded vector space, i.e

$$E = \bigoplus_{\alpha \in G} E_\alpha. \tag{6}$$

The family of all H-comodules forms a category $\mathcal{C} = \mathcal{M}^H$. The category $\mathcal{C}$ is braided monoidal. The monoidal operation in $\mathcal{C} = \mathcal{M}^H$ is given as the following tensor product of H-comodules

$$\rho_{E \otimes E} = (id \otimes m) \circ (id \otimes \tau \otimes id) \circ (\rho_E \otimes \rho_E), \tag{7}$$

where $\tau : H \otimes E \longrightarrow E \otimes H$ is the twist, $m : H \otimes H \longrightarrow H$ is the multiplication in H. The braid symmetry $\Psi \equiv \{\Psi_{U,V} : U \otimes V \longrightarrow V \otimes U; U, V \in Ob\mathcal{C}\}$ in $\mathcal{C}$ is defined by

$$\Psi_{U,V}(u \otimes v) = \lambda(\alpha, \beta) \, v \otimes u \tag{8}$$

for $u \in U_\alpha$, $v \in V_\beta$, λ is a bicharacter on the group G, i.e. a mapping $\lambda : G \times G \longrightarrow \mathbb{C} \setminus \{0\}$ such that

$$\lambda(\alpha + \beta, \gamma) = \lambda(\alpha, \gamma) \, \lambda(\beta, \gamma), \lambda(\alpha, \beta + \gamma) = \lambda(\alpha, \beta) \, \lambda(\alpha, \gamma). \tag{9}$$

for all $\alpha, \beta, \gamma \in G$. In this paper we restrict our attention to abelian grading group G and normalized bicharacters (called by Scheunert [11] a commutation factors on G) such that

$$\lambda(\alpha, \beta) \; \lambda(\beta, \alpha) = 1. \tag{10}$$

In this particular case the category $\mathcal{C}$ becomes symmetric.

3 Algebra of states

It is interesting that there exist an algebra $\mathcal{A}$ in the symmetric monoidal category $\mathcal{C}$ which is G-graded

$$\mathcal{A} = \bigoplus_{\alpha \in G} \mathcal{A}_\alpha, \quad \mathcal{A}_\alpha \, \mathcal{A}_\beta \subset \mathcal{A}_{\alpha+\beta}, \tag{11}$$

and λ-commutative

$$x_\alpha \; x_\beta = \lambda(\alpha, \beta) \; x_\beta \; x_\alpha \tag{12}$$

for homogeneous $x_\alpha \in \mathcal{A}_\alpha$, $x_\beta \in \mathcal{A}_\beta$. Note that the algebra $\mathcal{A}$ is also a coalgebra

$$\Delta x_\gamma = \sum_{\alpha+\beta=\gamma} \sum_i x_{\beta,i} \; x_{\alpha,i}. \tag{13}$$

Moreover, one can see that there is a graded Hopf algebra structure on $\mathcal{A}$, see [4]. We use here the so-called standard gradation for simplicity. In this gradation the algebra $\mathcal{A}$ is generated by x_i, $(i = 1, ..., N)$ such that

$$x_i \; x_j = \lambda_{ij} \; x_j \; x_i, \tag{14}$$

where the factor λ_{ij} is defined by the following formula

$$\lambda_{ij} := \lambda(\sigma_i, \sigma_j), \quad i, j = 0, 1, ..., N, \tag{15}$$

σ_i for $(i = 1, ..., N)$ are generators of G, $grade(x_i) = \sigma_i$, and it is natural to assume that $\sigma_0 \equiv e$, e is the neutral element in G. It is obvious that $\lambda_{ij} \in \mathbb{C} \setminus \{0\}$ for every $i, j = 1, ..., N$ and

$$\lambda_{ii} = \pm 1, \quad \lambda_{ij} \, \lambda_{ji} = 1, \quad \lambda_{0j} = \lambda_{i0} = \lambda_{00} = 1. \tag{16}$$

The set of numbers $\{\lambda_{ij} : i,j = 0,1,...,N\}$ is said to be *a relative sign* or *relative phase*. Every element x_α of the algebra $\mathcal{A}$ can be given in the following form

$$x_\alpha = x_1^{\alpha_1}, ..., x_N^{\alpha_N} \tag{17}$$

for $\alpha = \Sigma_{i=1}^N a_i \sigma_i$, $x_0 \equiv \mathbf{1}$, where $\mathbf{1}$ is the unit in $\mathcal{A}$. In physical interpretation every element x_α of $\mathcal{A}$ describe certain configuration of charged particle x coupled to our singular magnetic field. Generators x_i of $\mathcal{A}$ correspond to particle coupled to single magnetic flux at the point s_i in the Landau lowest lewel. The unit $\mathbf{1}$ of the algebra $\mathcal{A}$ describe the particle which is not coupled to the magnetic field. The monomial $x_i x_j$ describe a particle coupled to two magnetic fluxes at s_i and s_j simultaneously. It is obvious that $(x_i)^2$ corresponds to particle coupled to two fluxes at the same point s_i. A charged particle equipped with magnetic flux is said to be *a quasiparticle*. The particle coupled to two magnetic fluxes at two different point is understood as a system of two different quasiparticles. It follows from our above interpretation that the monomial $x_i x_j$ describe particle coupled to two magnetic fluxes, i. e. a system of two different quasiparticles x_i and x_j. We assume that quasiparticles are identical. If we exchange these quasiparticles one by another, then we obtain the system $x_j x_i$ which must be equivalent to $x_i x_j$. The only difference can be in phase. In fact the λ-commutativity (14) means that really the only difference is in the phase λ_{ij}. This also means that our quasiparticles have his own statistics. Let x_α be an arbitrary element of the algebra $\mathcal{A}$ of the form (17). If $\alpha_i = 0$ for certain i, then $x_i^0 = \mathbf{1}$, and we say that we have a quasihole at s_i. In this way an element $x_\alpha \in \mathcal{A}$ describe a system of quaisparticles and quasiholes.

It is known that every commutation factor λ on an arbitrary grading group G can be given in the following form

$$\lambda(\alpha, \beta) = (-1)^{(\alpha|\beta)} \, q^{<\alpha|\beta>}, \tag{18}$$

where $(-|-)$ is an integer-valued symmetric bi-form on G, and $< -|- >$ is a skew-symmetric integer-valued bi-form on G, q is some complex parameter [16]. Note that q in general is not a root of unity, but in the particular case when $G = Z_m \oplus ... \oplus Z_m$ q must be the m-th root of unity.

4 Commutation factor for composite bosons and fermions

Let us consider the grading group and commutation factor corresponding to our composite system of particle and magnetic fluxes. We use here the so-called standart gradation of Ref. [12]. In this gradation the grading group G is in general equal to $Z^N := Z \oplus ... \oplus Z$ (N-sumands), $N = 0, 1, ...$; $Z^0 \equiv \{e\}$ is the trivial group, and $Z^1 \equiv Z$ is the group of all integers. In physical interpretation N denote the number of singular points. Let us calculate explicite the relative phase λ_{ij} for our model. It follows immediately from formule (15) and (18) that we have the following general relation

$$\lambda_{ij} = \lambda(\sigma_i, \sigma_j) = (-1)^{A_{ij}} \, q^{\Omega_{ij}},$$

where $A_{ij} := (\sigma_i|\sigma_j)$, and $\Omega_{ij} :=< \sigma_i|\sigma_j >$ are integer-valued matrices such that $A_{ij} = A_{ji}$ and $\Omega_{ij} = -\Omega_{ji}$. The above relative phase contains two factors: $\lambda_{ij}^{st} := (-1)^{A_{ij}}$ and $\lambda_{ij}^{A-B} := q^{\Omega_{ij}}$. In our physical interpretation λ_{ij} describes the phase corresponding to interchanging of two qusiparticles x_i and x_j. Remember that quasiparticles x_i and x_j are in fact the same particle x but equipped with two fluxes. It is natural to assume that the first factor describes the statistics of the original particle x (i.e. the particle under consideration, without magnetic fluxes) and the second one - the Aharamov-Bohm phase for two interchanging quasiparticles. If our particle is electron, then we must assume that $\lambda_{ii}^{st} \equiv -1$ and $\lambda_{ij}^{st} \equiv 1$ for all $i \neq j$, This means that $A_{ij} = \delta_{ij}$ and we have $\lambda_{ij}^{st} := (-1)^{\delta_{ij}}$. For λ_{ij}^{A-B} we assume that

$$\lambda_{ij}^{A-B} := \exp\left[\pi i(\tfrac{e}{h}\Phi + 1) \right], \qquad (19)$$

for $i < j$, $e \neq 0$ is the electric charge, h is the Planck constant. For $i > j$ we have $\lambda_{ij}^{A-B} = (\lambda_{ji}^{A-B})^{-1}$, $\lambda_{ii}^{A-B} = 1$. It follows immediately from the above formula (19) that for

$$\Phi = N\Phi_0 = N\tfrac{h}{e}, \qquad (20)$$

where $N = 0, 1, 2, ...$ we have

$$\lambda_{ij}^{A-B} = \exp\left[\pi i(N + 1) \right] = \begin{cases} -1 \text{ for } N \text{ even} \\ +1 \text{ for } N \text{ odd} \end{cases}.$$

This means that we obtain $\lambda_{ij}^{A-B} = -(-1)^N$. Hence the relative phase λ_{ij} can be given in the form

$$\lambda_{ij} = -(-1)^{\delta_{ij}} q^{\Omega_{ij}}, \tag{21}$$

where $q := \exp(\pi i N)$, $\Omega_{ij} = -\Omega_{ji} = 1$, for $i \neq j$, $\Omega_{ii} = 0$, $i, j = 1, 2, ..., N$. This means that we have $\lambda_{ij} = \pm 1$ and the grading group $G = Z^N$ can be reduced to the following group

$$G \equiv Z_2^N := Z_2 \oplus ... \oplus Z_2 \quad (N \, sumands). \tag{22}$$

This also means that the charged particle can be coupled to one elementary magnetic flux Ψ_e at every singular point s_i and the filling factor is $v = \frac{1}{N}$. Let $\mathcal{A} \equiv \mathcal{A}_\lambda$ be a G-graded λ-commutative algebra, where G is given by the formula (22) and λ by (21). Here an arbitrary element x_α of $\mathcal{A}_\lambda$ can be given in the form (17), where $\alpha_i = 0$ or 1 for $i = 1, ..., N$. If $\alpha_i = 0$, then we have quasihole, if $\alpha_i = 1$, then we have quasiparticle. The number of quasiparticles is: $m = \Sigma_{i=1}^N \alpha_i$. Now we looking for the realization of the algebra $\mathcal{A}_\lambda$ in the tensor product $\Lambda_s \otimes C_N$ of two others algebras Λ_s and C_N. We give the following ansatz

$$x_i := \Theta_i \otimes e_i \tag{23}$$

for $i = 1, ..., N$. We assume that the algebra Λ_s is generated by $\Theta_1, ..., \Theta_N$ such that we have the following commutation relations

$$\Theta_i \, \Theta_j = -\lambda_{ij} \, \Theta_j \, \Theta_i. \tag{24}$$

For $i = j$ we obtain that $\Theta_i^2 = 0$. The algebra C_N is the Cliford algebra. It is generated by $e_1, ..., e_N$ such that we hahe the well-known relations

$$e_i \, e_j = -e_j \, e_i \quad \text{for} \quad i \neq j, \quad e_i^2 = 1 \tag{25}$$

For the multiplication in the algebra $\mathcal{A} \equiv \mathcal{A}_\lambda$ we have

$$x_i \, x_j = \Theta_i \, \Theta_j \otimes e_i e_j. \tag{26}$$

Now let us study the above realization of $\mathcal{A} \equiv \mathcal{A}_\lambda$ in more details. Observe that for even N we obtain

$$\Theta_i \, \Theta_j = \Theta_j \, \Theta_i, \quad \Theta_i^2 = 0. \tag{27}$$

It is interesting that such algebra can be represented by one grassmann variable Θ, $\Theta^2 = 0$ as follows

$$\Theta_i = (0, ..., \underset{\underset{\text{the i-th place}}{\uparrow}}{\Theta}, ..., 0). \tag{28}$$

We have

$$\Theta_i \Theta_j = (0, ..., \underset{\underset{\text{the i-th place}}{\uparrow}}{\Theta}, ..., \underset{\underset{\text{the j-th place}}{\uparrow}}{\Theta}, ..., 0) = \Theta_j \Theta_i. \tag{29}$$

and

$$\Theta_i^2 = (0, ..., \underset{\underset{\text{the i-th place}}{\uparrow}}{\Theta^2}, ..., 0) = 0. \tag{30}$$

For the quantum state $x_i x_j (i \neq j)$ representing particle coupled to magnetic fluxes at two different points we have

$$x_i \, x_j = (0, ..., \Theta, ..., \Theta, ..., 0) \otimes e_i \, e_j. \tag{31}$$

Now it is easy to see that the algebra $\mathcal{A}_\lambda$ for even N describes composite fermions. Let us consider the case of $N = 2$ in more details. In this case we have the following atates

$$x_1 = (\Theta, 0) \otimes e_1, \quad x_2 = (0, \Theta) \otimes e_2, \tag{32}$$

and

$$x_1 \, x_2 = (\Theta, \Theta) \otimes e_1 \, e_2. \tag{33}$$

The filling factor for all these states (32) and (33) is $v = \frac{1}{2}$. The states (32) contain quasiholes. Note that the state (33) does not contain quasiholes and hence it is unique state corresponding for which the magnetic field is completely compensated!

For odd N we have

$$\Theta_i \, \Theta_j = -\Theta_i \, \Theta_j \quad \text{for } i \neq j. \tag{34}$$

For $i = j$ we obtain the identity $\Theta_i \Theta_i = \Theta_i \Theta_i$. In this case the algebra $\mathcal{A}_\lambda$ can also be represented by the variable Θ such that we have

$$\Theta_i = (0, ..., \underset{\underset{\text{the i-th place}}{\uparrow}}{\Theta}, ..., 0).$$

(35)

We have

$$\Theta_i \Theta_j = (0, ..., \underset{\underset{\text{the i-th place}}{\uparrow}}{\Theta}, ..., \underset{\underset{\text{the j-th place}}{\uparrow}}{\Theta}, ..., 0) = -\Theta_j \Theta_i.$$

(36)

This means that

$$\Theta_i \, \Theta_j = 0$$

(37)

for all $i \neq j$. Observe that the quantum state $x_i x_j$ $(i \neq j)$ corresponding to particle coupled to magnetic fluxes at two different points disappear

$$x_i \, x_j = \Theta_i \, \Theta_j \otimes e_i \, e_j = 0,$$

(38)

and the state describing the particle coupled to a few fluxes at the same point is also impossible. In fact we have

$$x_i^2 = (0, ..., \Theta^2, ..., 0) \otimes e_i^2 = 0.$$

(39)

This means that the state x_i^2 is not equipped with a flux. Let us consider as an example the case of $N = 3$ i.e. the filling factor is $v = \frac{1}{3}$. In this case we have the states

$$x_1 = (\Theta, 0, 0) \otimes e_1, \quad x_2 = (0, \Theta, 0) \otimes e_2, \quad x_3 = (0, 0, \Theta) \otimes e_3$$

(40)

which contain two quasiholes. Observe that the following states

$$x_1 x_2, \quad x_1 x_3, \quad x_2 x_3.$$

(41)

which contain one quasihole and the state

$$x_1 x_2 x_3$$

(42)

which not contain quasiholes are impossiible. Hence in this case the single quasiparticle states with two quasiholes are possible! This also means that

the Landau lowest lewel is $\frac{1}{3}$-filled. In this way we obtain simple algebraic description of quantum states for particle in singular magnetic field which agree with known facts.

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